Distributed Emergent Model Reference Adaptive Control for Multi-Channel Linear Plant with Unknown Parameters

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***Abstract*— Accurate knowledge of plant parameters is essential for model-based control design. However, in cooperative control of a linear multi-channel plant, identifying such parameters is difficult because networked agents actuate a common plant through independent local input channels without knowledge of the global input channel. To avoid explicit system identification, this paper proposes a distributed model reference adaptive control (MRAC) framework for setpoint tracking of a linear multi-channel plant with unknown parameters. Each agent implements local reference dynamics that are coupled over a communication network, whose synchronized motion gives rise to an emergent reference model through the blended dynamics. The plant then tracks the resulting reference trajectory using distributed MRAC. We prove asymptotic output convergence to the setpoint and boundedness of all adaptive gains. Simulations show that the proposed method achieves adaptive tracking performance under parameter uncertainty.**

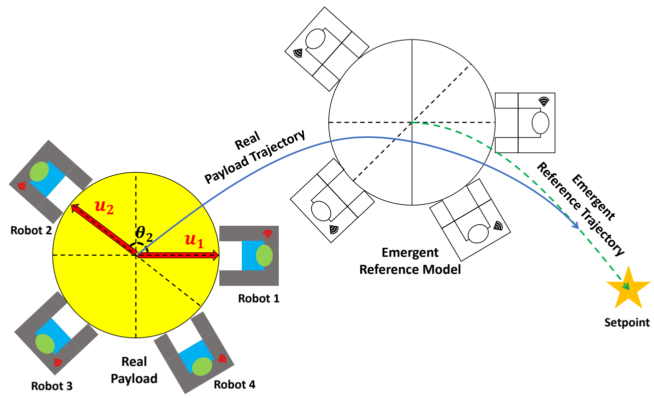

Fig. 1. Cooperative payload transportation and the proposed distributed control architecture. Each robot acts on the real payload through its own local input channel while running local reference dynamics. Through the blended dynamics scheme, these dynamics collectively generate an emergent reference model. The payload then tracks its reference trajectory via distributed MRAC.

I. INTRODUCTION

Distributed multi-channel control is a cooperative control framework in which many agents jointly control a common plant through independent local input channels. This framework is used in applications such as multi-agent object transportation [1], [2] and vibration suppression of large-scale flexible structures with distributed actuators [3]. However, these works assume that the designer knows the system parameters exactly. In practice, exact values of physical parameters are often unavailable. This calls for the development of a distributed adaptive control method that does not rely on exact parameter knowledge.

A motivating example is shown in Fig. 1. In this scenario, robot i knows only the angle θ_i at which it is attached to the payload, and applies a control input u_i along that direction as a pushing or pulling force. In addition, the robots communicate with neighboring robots over a communication network. While each robot knows the direction of its own local input channel, the actual effect of each robot's input on the payload is uncertain because of unknown parameters such as mass, damping, and actuator gains. Existing distributed methods [1], [4], [5] are not directly applicable in this setting because they require exact parameter knowledge.

Our approach leverages model reference adaptive control (MRAC) [6], which drives the plant to follow a reference model without explicit parameter identification. Although some studies have applied MRAC to multi-agent systems [7], [8], they focus on consensus problems rather than the multi-channel setting. In our setting, each agent uses local channel information to implement local reference dynamics and exchanges its local reference states with neighboring agents over the communication network. Through the blended dynamics scheme [9], these local reference dynamics, coupled via neighbor communication, generate what we call the *emergent reference model*: a virtual reference model that is neither designed by nor available to any individual agent, but arises collectively from their interaction. The plant then tracks the resulting reference trajectory via distributed MRAC, using the local reference states generated by the local reference dynamics.

Several studies have addressed related cooperative control problems, but they consider different settings. Observer-based distributed control methods [4], [5] require each agent to know all local input channels, which may be impractical in decentralized settings. The decentralized multi-channel control scheme in [1] removes this requirement, but assumes exact parameter knowledge. Meanwhile, [10] addresses object transportation with unknown physical parameters without inter-agent communication, but uses full three-dimensional vector inputs rather than the multi-channel setting considered here. Therefore, the proposed framework builds on the decentralized multi-channel scheme of [1] while incorporating adaptive compensation for parameter uncertainty.

The main contributions of this paper are as follows. First,

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we show that local reference dynamics designed without knowledge of the global input channel generate, through neighbor communication, an emergent reference model for setpoint regulation. Second, we integrate this emergent reference model with distributed MRAC to achieve adaptive setpoint tracking without global parameter knowledge. Third, we prove asymptotic output convergence and bounded adaptive gains, and demonstrate the adaptive tracking performance through simulations.

Notation: Throughout the paper, for a symmetric matrix A, $\lambda_i(A)$ denotes the i-th smallest eigenvalue, while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote its minimum and maximum eigenvalues, respectively. In addition, $A \succ 0$ ($A \succeq 0$) means that A is positive definite (positive semidefinite). For $P \succ 0$, $P^{1/2}$ denotes its square root such that $(P^{1/2})^2 = P$. The symbols $\mathrm{sgn}(\cdot)$, $\mathrm{tr}(\cdot)$, $\otimes$, $|\cdot|$, $\|\cdot\|$, and $\|\cdot\|_F$ denote the sign, trace, Kronecker product, Euclidean norm, induced 2-norm, and Frobenius norm, respectively. For matrices $A_1,\ldots,A_N$, the notation $\mathrm{diag}_N(A_i)$ denotes the block diagonal matrix $\mathrm{diag}(A_1,\ldots,A_N)$. The matrix I_n is the $n\times n$ identity matrix, $1_n \in \mathbb{R}^n$ represents the vector whose elements are all 1, $0_{n\times m} \in \mathbb{R}^{n\times m}$ is the zero matrix, and $\mathscr{L}_1$ is the set of absolutely integrable functions on $[0,\infty)$. For the communication network, let $\mathscr{G} = (\mathscr{N},\mathscr{E})$ denote a graph, where $\mathscr{N} = \{1,\ldots,N\}$ is the node set and $\mathscr{E} \subseteq \mathscr{N}\times\mathscr{N}$ is the edge set. The neighbor set of agent i is defined as $\mathscr{N}_i := \{j \in \mathscr{N} : (i,j) \in \mathscr{E}\}$.

II. PROBLEM FORMULATION

Motivated by the cooperative payload transportation scenario in Fig. 1, we consider a linear time-invariant (LTI) multi-channel plant actuated by N independent control agents through local input channels:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} b_i u_i(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $A \in \mathbb{R}^{n\times n}$ is the plant matrix, and agent $i \in \{1,\ldots,N\}$ applies its local input $u_i(t) \in \mathbb{R}^{m_i}$ only through the input channel matrix $b_i \in \mathbb{R}^{n\times m_i}$. The stacked input matrix and input vector are defined as $B = [b_1 \cdots b_N] \in \mathbb{R}^{n\times m}$ and $u(t) = [u_1^\top(t) \cdots u_N^\top(t)]^\top \in \mathbb{R}^m$, respectively, where $m = \sum_{i=1}^{N} m_i$. The output is given by $y(t) = Cx(t) \in \mathbb{R}^p$, where $C \in \mathbb{R}^{p\times n}$, and the control objective is to regulate $y(t)$ to a constant setpoint $r \in \mathbb{R}^p$.

The following assumptions specify the plant structure, the information available to each agent, and the communication environment. The subsequent remark explains their practical relevance.

Assumption 1 (Parameter uncertainty and local information)**.** The plant matrix A is unknown. The input matrix satisfies $B = B_o D$, where $B_o = [b_{o,1} \cdots b_{o,N}] \in \mathbb{R}^{n\times m}$ is the nominal input matrix, and agent i knows only its own nominal input channel $b_{o,i} \in \mathbb{R}^{n\times m_i}$. Moreover, the gain matrix $D = \mathrm{diag}_N(d_i I_{m_i}) \in \mathbb{R}^{m\times m}$ is block diagonal, where each d_i is an unknown nonzero scalar, and agent i knows only the sign of its own d_i.

Assumption 2 (Controllability)**.** The pair (A,B) is controllable, whereas the pair (A,b_i) need not be controllable.

Assumption 3 (Measurement and reference information)**.** Each agent can measure x and knows C and r.

Assumption 4 (Setpoint regulation condition)**.** The plant (A,B,C) has no invariant zero at the origin; equivalently, $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ has full row rank.

Assumption 5 (Communication graph)**.** The communication graph is an undirected, connected, unit-weight graph.

Remark 1. In many practical multi-channel systems, the structural form of the input channels is available from manufacturer specifications even when the exact physical coefficients are uncertain. In the example shown in Fig. 1, robot i can construct its nominal input channel $b_{o,i}$ from the measured θ_i as detailed in Section V, while uncertain payload mass and actuator gains appear through the unknown channel gain d_i, motivating the factorization $B = B_o D$. For the class of systems considered here, Assumption 4 can likewise be verified from the known system structure without full knowledge of the exact physical coefficients. Note that this condition requires $p \le m$, which is satisfied when sufficiently many agents are deployed.

Under Assumptions 1–5, the problem is to design a distributed adaptive control law that asymptotically regulates the output $y(t)$ to the setpoint r despite uncertainty in the plant matrix and input channel gains, while all adaptive gains remain bounded.

III. DISTRIBUTED EMERGENT REFERENCE MODEL GENERATION

For adaptive control, the controller requires a reference model that specifies the desired plant evolution. This section identifies the desired form of the reference model to be generated through the local reference dynamics for setpoint regulation. Then, we design such local dynamics that each agent can implement without knowledge of the global input channel. Using the blended dynamics scheme [9], we show that the coupled local reference dynamics give rise to an emergent reference model and its reference trajectory.

A. Desired Reference Model and Matching Condition

We first investigate the form of the reference model to be generated. For constant setpoint tracking, we adopt a proportional-integral (PI) augmented structure. That is, the reference model should take the closed-loop form

$$\dot{x}_g(t) = (\bar{A}_M - \bar{B}_M G(t))x_g(t) + Fr, \tag{2}$$

where $x_g \in \mathbb{R}^{\bar{n}}$ with $\bar{n} := n + p$ is the PI-augmented reference-model state. The block matrices are

$$\bar{A}_M := \begin{bmatrix} A_M & 0 \\ -C & 0 \end{bmatrix}, \quad \bar{B}_M := \begin{bmatrix} B_M \\ 0_{p\times m} \end{bmatrix}, \quad F := \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix},$$

where $A_M \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix shared by all agents, and $B_M \in \mathbb{R}^{n\times m}$ is the input matrix whose local channels will be specified shortly. By the block structures of $\bar{A}_M$, $\bar{B}_M$, and F, the first n components of x_g correspond to the plant state component, and the last p components correspond to the integrator state component. The feedback gain $G(t) \in \mathbb{R}^{m\times\bar{n}}$ is necessary to stabilize the PI-augmented pair $(\bar{A}_M, \bar{B}_M)$, although A_M is Hurwitz. Section III-B shows how the agents generate this gain in a distributed manner using local channel information. Note that (2) merely specifies the desired form, and both B_M and $G(t)$ will be generated in a distributed manner by the agents.

For the plant to track the desired reference model (2) via MRAC, the plant and the reference model must satisfy the following matching condition [6]: there exist $K^* \in \mathbb{R}^{m\times n}$ and a positive definite matrix $L^* \in \mathbb{R}^{m\times m}$ such that

$$A - BK^* = A_M, \quad BL^* = B_M. \tag{3}$$

Since each $b_{o,i}$ is available only as local information to agent i and no centralized coordinator collects them, no single agent can determine B or construct B_M. Instead, each agent independently constructs its local reference input channel using its own local information as follows:

$$b_{M,i} := \sigma_i b_{o,i}\,\mathrm{sgn}(d_i), \tag{4}$$

where $\sigma_i > 0$ is a positive scalar chosen by agent i, and $B_M = [b_{M,1} \cdots b_{M,N}] \in \mathbb{R}^{n\times m}$. Each agent then forms the local augmented channel corresponding to $\bar{B}_M = [\bar{b}_{M,1} \cdots \bar{b}_{M,N}]$ as

$$\bar{b}_{M,i} := \begin{bmatrix} b_{M,i} \\ 0_{p\times m_i} \end{bmatrix} \in \mathbb{R}^{\bar{n}\times m_i}. \tag{5}$$

By construction (4), the input matching condition $BL^* = B_M$ automatically holds with the feedforward gain matrix:

$$L^* = D^{-1}\,\mathrm{diag}_N(\sigma_i\,\mathrm{sgn}(d_i) I_{m_i}) \succ 0. \tag{6}$$

Consequently, L^* is positive definite and diagonal. Note that no agent needs to compute (6). It only certifies that the construction rule (4) induces the matching condition.

Now, the following assumption imposes the remaining part of the matching condition.

Assumption 6 (Matching feasibility)**.** For the Hurwitz matrix A_M, there exists $K^* \in \mathbb{R}^{m\times n}$ satisfying $A - BK^* = A_M$.

Remark 2. Under Assumption 6 and $B_M = BL^*$ with invertible L^*, Assumption 4 is equivalent to the full-row-rank condition on

$$\begin{bmatrix} A_M & B_M \\ -C & 0 \end{bmatrix}. \tag{7}$$

Remark 3. Section V illustrates, through the payload-transportation example, that the local channels $b_{M,i}$ and a matrix A_M satisfying Assumption 6 can be designed from the known kinematic structure, without knowledge of the mass, damping, or actuator gains.

B. Local Reference Dynamics for Emergent Reference Model

Since each agent has access only to its $\bar{b}_{M,i}$, we design the following local reference dynamics whose synchronized motion gives rise to the desired reference model (2):

$$\dot{\hat{z}}_{g,i}(t) = -k_g \sum_{j\in\mathcal{N}_i} (\hat{x}_{g,j}(t) - \hat{x}_{g,i}(t)), \tag{8a}$$

$$\begin{aligned}\dot{\hat{x}}_{g,i}(t) = {}& \bar{A}_M \hat{x}_{g,i}(t) + N\bar{b}_{M,i} u_{r,i}(t) + Fr \\ &+ k_g \sum_{j\in\mathcal{N}_i} (\hat{x}_{g,j}(t) - \hat{x}_{g,i}(t)) + k_g \sum_{j\in\mathcal{N}_i} (\hat{z}_{g,j}(t) - \hat{z}_{g,i}(t)),\end{aligned} \tag{8b}$$

$$u_{r,i}(t) = -G_i(t)\hat{x}_{g,i}(t), \tag{8c}$$

where $\hat{x}_{g,i} := [\hat{x}_{r,i}^\top\ \hat{w}_{r,i}^\top]^\top \in \mathbb{R}^{\bar{n}}$ is the local augmented reference state containing the local reference state $\hat{x}_{r,i} \in \mathbb{R}^n$ and $\hat{w}_{r,i} \in \mathbb{R}^p$. The signal $u_{r,i}$ is the local reference input, $\hat{z}_{g,i} \in \mathbb{R}^{\bar{n}}$ is the integral coupling state, $k_g > 0$ is the coupling gain, and $G_i(t) \in \mathbb{R}^{m_i\times\bar{n}}$ is the local feedback gain.

Let $G_i(t)$ converge to $G_i^* \in \mathbb{R}^{m_i\times\bar{n}}$ exponentially, and define $G^* := [G_1^{*\top} \cdots G_N^{*\top}]^\top \in \mathbb{R}^{m\times\bar{n}}$. We assume that G^* makes $\bar{A}_M - \bar{B}_M G^*$ Hurwitz. Distributed construction of local gains achieving this property is given after Lemma 1. To simplify the algorithm, we assume that all agents already know the network size N. This assumption can be relaxed by employing distributed network size estimation techniques such as [11].

Remark 4. The local reference dynamics (8) directly incorporates the PI-augmented structure of (2), thereby embedding the regulation objective into the dynamics. A simpler alternative at first glance is to first compute an equilibrium pair (x^*, u^*) satisfying $A_M x^* + B_M u^* = 0$ and $Cx^* = r$, and then translate the coordinates so that the equilibrium associated with r becomes the origin. However, this is not straightforward in the present distributed setting, because computing (x^*, u^*) requires global knowledge of B_M, whereas each agent knows only its own local channel. Thus, the PI-augmented structure avoids explicit distributed computation of the equilibrium.

To explain why (8) generates the emergent reference model, we adopt the blended dynamics concept [9]. Consider a PI-coupled network of the form

$$\begin{aligned}\dot{x}_i(t) &= f_i(x_i(t)) + k \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)) + k \sum_{j\in\mathcal{N}_i} (z_j(t) - z_i(t)),\\ \dot{z}_i(t) &= -k \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)),\end{aligned}$$

where f_i denotes the local vector field of agent i. Under suitable conditions, the local states are driven close to synchronized motion, and the resulting collective behavior can be understood through a lower-dimensional averaged dynamics, called the blended dynamics:

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(s(t)).$$

This reduced dynamics is obtained by restricting the network motion to synchronized trajectories, where $s(t)$ denotes the blended state. In this viewpoint, the PI coupling terms drive the agents toward synchronization, while the integral coupling state z_i mainly affects the synchronization transient and does not appear explicitly in the resulting blended dynamics.

Applying this mechanism to (8), we stack the local reference inputs and local feedback gains as

$$u_r(t) := [u_{r,1}^\top(t) \ \cdots \ u_{r,N}^\top(t)]^\top, \quad G(t) := [G_1^\top(t) \ \cdots \ G_N^\top(t)]^\top.$$

Then, under the synchronized motion $\hat{x}_{g,i}(t) \equiv s(t)$ for all i, the corresponding blended dynamics is

$$\dot{s}(t) = (\bar{A}_M - \bar{B}_M G(t))s(t) + Fr, \tag{9}$$

which we call the *emergent reference model*. We call the first n components of $s(t)$, associated with the plant state, the *emergent reference trajectory*.

If $G_i(t) \equiv G_i^*$, (9) becomes

$$\dot{s}(t) = (\bar{A}_M - \bar{B}_M G^*)s(t) + Fr, \tag{10}$$

which has the same form as the desired reference model (2). Since $\bar{A}_M - \bar{B}_M G^*$ is Hurwitz, (10) admits a unique exponentially stable equilibrium x_g^*.

We also define the network average of the local augmented reference states as

$$\rho_{\text{avg}}(t) := \frac{1}{N}\sum_{i=1}^{N} \hat{x}_{g,i}(t) =: \begin{bmatrix} x_r^\top(t) & w_r^\top(t) \end{bmatrix}^\top, \tag{11}$$

where $x_r(t)$ and $w_r(t)$ denote the plant state and integrator state components of $\rho_{\text{avg}}(t)$, respectively. Note that $\rho_{\text{avg}}(t)$ is not a signal available to any individual agent but a collective variable used only for analysis. The following lemma shows that, for a sufficiently large coupling gain, all $\hat{x}_{g,i}(t)$ and $\rho_{\text{avg}}(t)$ converge exponentially to x_g^* even when $G_i(t)$ is time-varying.

Lemma 1. Suppose Assumptions 1–6 hold, and each local gain $G_i(t)$ converges to G_i^* exponentially for all $i \in \{1,\ldots,N\}$. Then there exists $k_g^* > 0$ such that, for all $k_g > k_g^*$, the $\hat{x}_{g,i}(t)$ in (8) converges exponentially to the unique equilibrium x_g^* in (10) and $\hat{z}_{g,i}(t)$ remains bounded for all i. Moreover, $\rho_{\text{avg}}(t)$ converges to x_g^* exponentially, so that $x_r(t)$ converges to x_r^* exponentially, where x_r^* denotes the plant state component of x_g^*, i.e., its first n components.

Proof. The complete proof is provided in the Appendix.

To complete the local reference dynamics construction, it remains to synthesize the local gains $G_i(t)$ in a distributed manner. For this purpose, we employ the distributed Bass algorithm [1]:

$$\begin{aligned}\dot{X}_i(t) &= \alpha_b \mathscr{F}_i(X_i(t)) \\ &\quad + k_b \sum_{j\in\mathscr{N}_i}(X_j(t) - X_i(t)) + k_b \sum_{j\in\mathscr{N}_i}(Z_j(t) - Z_i(t)), \\ \dot{Z}_i(t) &= -k_b \sum_{j\in\mathscr{N}_i}(X_j(t) - X_i(t)),\end{aligned}$$

where $X_i(t), Z_i(t) \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\alpha_b, k_b > 0$, $X_i(0) \succ 0$, and

$$\mathscr{F}_i(X) := -(\bar{A}_M + \beta I_{\bar{n}})X - X(\bar{A}_M + \beta I_{\bar{n}})^\top + 2\bar{b}_{M,i}\bar{b}_{M,i}^\top.$$

We choose $\beta > 0$ sufficiently large so that $-(\bar{A}_M + \beta I_{\bar{n}})$ is Hurwitz.

The distributed Bass algorithm adopts the same PI consensus coupling structure as (8), and the associated blended dynamics is the differential Lyapunov equation

$$\dot{S} = \alpha_b\Big[-(\bar{A}_M + \beta I_{\bar{n}})S - S(\bar{A}_M + \beta I_{\bar{n}})^\top + \frac{2}{N}\bar{B}_M\bar{B}_M^\top\Big],$$

where $S \in \mathbb{R}^{\bar{n}\times\bar{n}}$ is the blended state of X_i. Since L^* is invertible and the matching conditions $A_M = A - BK^*$ and $B_M = BL^*$ hold, the controllability of (A,B) implies that of (A_M, B_M) via state feedback and an invertible input transformation. Since the matrix (7) has full row rank, the PBH test confirms that $(\bar{A}_M, \bar{B}_M)$ is controllable. Therefore, the differential Lyapunov equation has a unique globally exponentially stable equilibrium $X^* \succ 0$, and by [1, Theorem 2], $X_i(t)$ converges to X^* exponentially for all i.

Let $P_s := NX^*$. Then $P_s \succ 0$ is the unique solution to

$$0 = -(\bar{A}_M + \beta I_{\bar{n}})P_s - P_s(\bar{A}_M + \beta I_{\bar{n}})^\top + 2\bar{B}_M\bar{B}_M^\top. \tag{12}$$

With this result, we synthesize the local feedback gain as

$$G_i(t) = \frac{1}{N}\bar{b}_{M,i}^\top X_i^{-1}(t). \tag{13}$$

See [1] for detailed techniques that avoid singular inversions during transients while preserving convergence to the identical limit gain. As $X_i(t) \to X^*$, the gain $G_i(t)$ converges exponentially to

$$G_i^* := \frac{1}{N}\bar{b}_{M,i}^\top (X^*)^{-1} = \bar{b}_{M,i}^\top P_s^{-1}. \tag{14}$$

Accordingly, the stacked limit gain is $G^* = [G_1^{*\top} \ \cdots \ G_N^{*\top}]^\top = \bar{B}_M^\top P_s^{-1}$. By the classical Bass stabilization result [12], this gain makes $\bar{A}_M - \bar{B}_M G^*$ Hurwitz, thereby satisfying the conditions of Lemma 1.

Since each agent has access to a local reference state $\hat{x}_{r,i}(t)$, the next section develops an adaptive controller that uses these local reference states to drive the plant to the setpoint.

IV. DISTRIBUTED ADAPTIVE CONTROL

A conventional centralized adaptive law would require global reference signals that are not available to any individual agent. Instead, each agent implements the following adaptive law using its locally available reference signals $\hat{x}_{r,i}(t)$ and $u_{r,i}(t)$. We modify standard MRAC [6] in a distributed manner:

$$\dot{K}_i(t) = \alpha_{K,i} b_{M,i}^\top P_M \big(x(t) - \hat{x}_{r,i}(t)\big) x^\top(t), \tag{15a}$$

$$\dot{L}_i(t) = -\alpha_{L,i} b_{M,i}^\top P_M \big(x(t) - \hat{x}_{r,i}(t)\big) u_{r,i}^\top(t), \tag{15b}$$

$$u_i(t) = -K_i(t)x(t) + L_i(t)u_{r,i}(t), \tag{15c}$$

where $K_i(t) \in \mathbb{R}^{m_i\times n}$ and $L_i(t) \in \mathbb{R}^{m_i\times m_i}$ denote the local adaptive gains, $\alpha_{K,i} > 0$ and $\alpha_{L,i} > 0$ are positive adaptation rates, and $P_M \succ 0$ is the unique solution to the Lyapunov equation $A_M^\top P_M + P_M A_M = -Q_M$ for a given $Q_M \succ 0$.

Theorem 1. Suppose Assumptions 1–6 hold, and let $\hat{x}_{r,i}$ and $u_{r,i}$ be generated by the local reference dynamics (8a)–(8c) under the conditions of Lemma 1. Consider the plant (1) with the adaptive control law (15a)–(15c). Then, the output $y(t) = Cx(t)$ asymptotically converges to the setpoint r, and all adaptive gains $K_i(t)$ and $L_i(t)$ remain bounded.

Proof. Define the stacked gain matrices

$$K(t) := [K_1^\top(t) \cdots K_N^\top(t)]^\top, \quad L(t) := \mathrm{diag}_N(L_i(t)).$$

By stacking the local inputs $u_i(t)$ from (15c) and the local reference inputs $u_{r,i}(t)$, the global plant input can be written as

$$u(t) = -K(t)x(t) + L(t)u_r(t),$$

where $u_r(t) := [u_{r,1}^\top(t) \cdots u_{r,N}^\top(t)]^\top$.

Now, define the tracking, local reference state, and adaptive gain errors as

$$\begin{aligned}
&e(t) := x(t) - x_r(t),\\
&e_{r,i} := x_r(t) - \hat{x}_{r,i}(t), \quad e_r(t) := [e_{r,1}^\top(t) \cdots e_{r,N}^\top(t)]^\top,\\
&\widetilde{K}(t) := K(t) - K^*, \quad \widetilde{L}(t) := L(t) - L^* = \mathrm{diag}_N(L_i(t) - L_i^*),
\end{aligned}$$

where K^* is defined in Assumption 6, L^* is given by (6), L_i^* denotes the i-th diagonal block of L^*, and $\widetilde{L}_i(t) := L_i(t) - L_i^*$.

Then, we can express

$$\begin{aligned}
&u(t) = -(K^* + \widetilde{K}(t))x(t) + (L^* + \widetilde{L}(t))u_r(t),\\
&\dot{\widetilde{K}}(t) = \dot{K}(t) = \alpha_K B_M^\top P_M e(t) x^\top(t) + \Delta_K(t),\\
&\dot{\widetilde{L}}(t) = \dot{L}(t) = \mathrm{diag}_N(-\alpha_{L,i} b_{M,i}^\top P_M e(t) u_{r,i}^\top(t) + \Delta_{L,i}(t)),
\end{aligned}$$

where $\alpha_K := \mathrm{diag}_N(\alpha_{K,i} I_{m_i})$,

$$\Delta_K(t) := \begin{bmatrix} \alpha_{K,1} b_{M,1}^\top P_M e_{r,1}(t) x^\top(t) \\ \vdots \\ \alpha_{K,N} b_{M,N}^\top P_M e_{r,N}(t) x^\top(t) \end{bmatrix},$$

and $\Delta_{L,i}(t) := -\alpha_{L,i} b_{M,i}^\top P_M e_{r,i}(t) u_{r,i}^\top(t)$. These perturbation terms arise from using $\hat{x}_{r,i}$ instead of x_r. Using these notations, the error dynamics become

$$\dot{e}(t) = A_M e(t) + B_M (L^*)^{-1}(-\widetilde{K}(t)x(t) + \widetilde{L}(t)u_r(t)) \quad (16)$$

by using both identities in the matching condition (3).

Now, set the Lyapunov function

$$V(e, \widetilde{K}, \widetilde{L}) = \frac{1}{2} e^\top P_M e + \frac{1}{2} \mathrm{tr}\left(\widetilde{K}^\top \Omega_K \widetilde{K}\right) + \frac{1}{2} \sum_{i=1}^N \mathrm{tr}\left(\widetilde{L}_i^\top \Omega_{L,i} \widetilde{L}_i\right)$$

where $\Omega_K := (\alpha_K L^*)^{-1}$ and $\Omega_{L,i} := (\alpha_{L,i} L_i^*)^{-1}$. Taking the derivative of V and using $BL^* = B_M$, $\Omega_K \alpha_K = (L^*)^{-1}$, and $\Omega_{L,i}\alpha_{L,i} = (L_i^*)^{-1}$, the $\widetilde{K}$ and $\widetilde{L}$ terms cancel by construction. Hence,

$$\dot{V} = -\frac{1}{2} e^\top Q_M e + \mathrm{tr}(\widetilde{K}^\top \Omega_K \Delta_K) + \sum_{i=1}^N \mathrm{tr}\left(\widetilde{L}_i^\top \Omega_{L,i} \Delta_{L,i}\right). \quad (17)$$

Applying the Cauchy–Schwarz inequality to the trace terms and using $\|\Omega_K^{1/2}\widetilde{K}\|_F \le \sqrt{2V}$ and $\|\Omega_{L,i}^{1/2}\widetilde{L}_i\|_F \le \sqrt{2V}$, which follow from the definition of V, we obtain

$$\dot{V} \le -\tfrac{1}{2} e^\top Q_M e + \sqrt{2V}\, \Delta_{KL}, \quad (18)$$

where $\Delta_{KL}(t) := \|\Omega_K^{1/2}\Delta_K(t)\|_F + \sum_{i=1}^N \|\Omega_{L,i}^{1/2}\Delta_{L,i}(t)\|_F$.

To bound this perturbation term, define $\delta_e(t) := \max_{i \in \mathcal{N}} |e_{r,i}(t)|$. By Lemma 1, both $x_r(t) - x_r^*$ and $\hat{x}_{r,i}(t) - x_r^*$ converge to zero exponentially. Hence, $e_{r,i}(t) = x_r(t) - \hat{x}_{r,i}(t)$ also converges to zero exponentially. It follows that δ_e converges to zero exponentially and is an $\mathscr{L}_1$ function.

Next, by Lemma 1, $\hat{x}_{g,i}(t)$ is bounded for all i. Moreover, since $G_i(t)$ converges exponentially to G_i^*, each $G_i(t)$ is also bounded. Hence, $u_{r,i}(t) = -G_i(t)\hat{x}_{g,i}(t)$ is bounded for all i, and therefore the stacked signal $u_r(t)$ is bounded. By (8) and (11), averaging over i cancels the coupling terms, yielding $\dot{x}_r(t) = A_M x_r(t) + B_M u_r(t)$. It follows that $x_r(t)$ is bounded because A_M is Hurwitz and u_r is bounded.

Moreover, each perturbation term is proportional to $e_{r,i}$ and multiplied by x or $u_{r,i}$, respectively. Since $u_{r,i}$ is bounded, there exists $c_p > 0$ such that

$$\Delta_{KL} \le c_p\, \delta_e \big(1 + |x|\big). \quad (19)$$

On the other hand, from $e = x - x_r$ and $V \ge (e^\top P_M e)/2 \ge (\lambda_{\min}(P_M)|e|^2)/2$, we have

$$|x| \le |x_r| + |e| \le c_r + \sqrt{\frac{2V}{\lambda_{\min}(P_M)}},$$

where c_r is the bound of $|x_r|$. Substituting this into (19) yields $\Delta_{KL} \le c_s \delta_e \big(1 + \sqrt{V}\big)$ for some $c_s > 0$. Finally, the perturbation term in (18) satisfies

$$\sqrt{2V}\, \Delta_{KL} \le c_s \delta_e \big(\sqrt{2V} + \sqrt{2}V\big) \le c_V\, \delta_e (1 + V)$$

for some $c_V > 0$. Hence, (18) gives $\dot{V} \le c_V\, \delta_e(1 + V)$.

Applying the Gronwall–Bellman inequality yields

$$1 + V(t) \le (1 + V(0)) \exp\left(c_V \int_0^t \delta_e(s)\, ds\right).$$

Because $\delta_e \in \mathscr{L}_1$, $V(t)$ is bounded for all $t \ge 0$. Consequently, $e(t)$, $\widetilde{K}(t)$, and $\widetilde{L}(t)$ are bounded, verifying that all adaptive gains remain bounded. Moreover, $x = x_r + e$ is bounded. Then, from (19) and the fact that δ_e decays exponentially, we can conclude that $\Delta_{KL} \in \mathscr{L}_1$ and Δ_{KL} converges to zero.

Integrating (18) over $[0, T]$ yields

$$\begin{aligned}
\frac{1}{2} \int_0^T e^\top(s) Q_M e(s)\, ds \le & V(0) - V(T)\\
& + \int_0^T \sqrt{2V(s)}\, \Delta_{KL}(s)\, ds.
\end{aligned}$$

Since V is bounded and $\Delta_{KL} \in \mathscr{L}_1$, the right-hand side remains bounded uniformly in T. Hence,

$$\int_0^\infty e^\top(s) Q_M e(s)\, ds$$

is finite. Therefore, from the error dynamics (16), all terms on the right-hand side are bounded, so $\dot{e}$ is bounded and hence $e(t)$ is uniformly continuous. Using Barbalat's lemma, $e(t)$ converges to 0. Then, Lemma 1 implies $x_r(t)$ converges to x_r^*, and the equilibrium relation $(\bar{A}_M - \bar{B}_M G^*)x_g^* + Fr = 0$ yields $Cx_r^* = r$. Since $e(t) \to 0$, we conclude that $y(t) = Cx(t) \to r$. $\square$

V. SIMULATION

In this section, we demonstrate the proposed distributed MRAC framework using the payload-transportation scenario illustrated in Fig. 1. The simulations show both the synchronization of the local reference states toward the emergent reference trajectory (Lemma 1) and the successful setpoint tracking of the adaptive controller under parameter uncertainty (Theorem 1). Let $x := [\mathrm{p}_x \;\; \mathrm{p}_y \;\; \mathrm{v}_x \;\; \mathrm{v}_y]^\top \in \mathbb{R}^4$ denote the position–velocity state. The plant model is

$$A = \begin{bmatrix} 0_{2\times 2} & I_2 \\ 0_{2\times 2} & -\frac{\mathscr{D}}{m_L} \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ 0 \\ \frac{\gamma_i}{m_L}\cos\theta_i \\ \frac{\gamma_i}{m_L}\sin\theta_i \end{bmatrix}, \quad C = \begin{bmatrix} I_2 & 0_{2\times 2} \end{bmatrix},$$

where m_L is the unknown payload mass, $\gamma_i > 0$ is the unknown actuator gain of robot i, and $\mathscr{D} \in \mathbb{R}^{2\times 2}$ is the unknown damping matrix. The setpoint is $r = [6,3]^\top$ m with the payload initially at the origin.

We consider $N = 4$ robots with attachment angles $\theta_i \in \{0, 6\pi/9, 10\pi/9, 14\pi/9\}$ rad, communicating over a ring graph. To model direction-dependent damping, we set

$$\mathscr{D} = \begin{bmatrix} 80 & -25 \\ -25 & 130 \end{bmatrix} \mathrm{N\cdot s/m}. \tag{20}$$

Following the proposed local channel construction rule, each agent chooses $\sigma_i = 1$. Since $d_i = \gamma_i/m_L > 0$ in this example, $\mathrm{sgn}(d_i) = 1$, and hence $b_{M,i} = b_{o,i} = \begin{bmatrix} 0 & 0 & \cos\theta_i & \sin\theta_i \end{bmatrix}^\top$. The reference model matrix

$$A_M = \begin{bmatrix} 0_{2\times 2} & I_2 \\ -I_2 & -2I_2 \end{bmatrix} \tag{21}$$

is designed to satisfy Assumption 6 based on the known kinematic structure and does not require knowledge of m_L, $\mathscr{D}$, or γ_i. With this choice, the overall simulation setup satisfies Assumptions 1 through 6.

We compare the proposed adaptive method with two fixed-gain baselines constructed using the method proposed in [1]: one designed with exact parameters and the other based on nominal parameters. For the nominal parameter baseline, we use the matrices A_M and B_M from the reference model. All methods use the distributed Bass algorithm with $\alpha_b = 5$ and $k_b = 10$. The nominal parameter baseline and the proposed method use $\beta = 3$, whereas the exact parameter baseline uses $\beta = 150$, which is tuned using the exact plant parameters. For the proposed method, we additionally set $k_g = 20$, $\alpha_{K,i} = \alpha_{L,i} = 1$, and $Q_M = 30I_4$. To illustrate adaptability, we run 100 Monte Carlo simulations only for the proposed method, with m_L and γ_i drawn from the uniform distribution on $[0.5, 3]$, while the two fixed-gain baselines are shown at the nominal operating point $m_L = 2$ kg and $\gamma_i = 0.7$. Each run lasts 10 s. The results are shown in Fig. 2.

In Fig. 2, the red dashed lines labeled as *Adaptive* represent the 100 payload trajectories under the proposed adaptive method. Despite variations in m_L and γ_i, the controller transports the payload to the setpoint. The blue solid line labeled as *Exact* shows the exact parameter baseline, which reaches

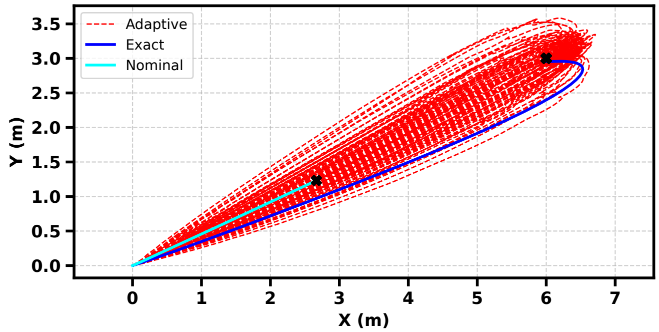

Fig. 2. Trajectories of the payload position simulated for 10 seconds. Red dashed: proposed adaptive method applied under 100 different combinations of mass and input scaling factors; Blue solid: fixed-gain designed with the exact model; Cyan solid: fixed-gain designed with a nominal model. Black crosses denote the terminal positions of the payload.

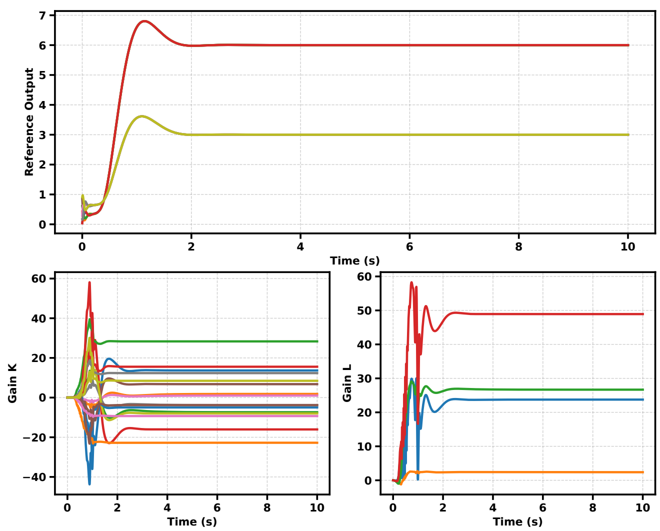

Fig. 3. Outputs of the local reference dynamics $C\hat{x}_{r,i}(t)$ for $i = 1, \ldots, 4$ (top), all entries of the adaptive row gains $K_i(t) \in \mathbb{R}^{1\times 4}$ (bottom left), and the scalar gains $L_i(t) \in \mathbb{R}$ (bottom right) during simulation. Top-panel trajectories overlap after synchronization.

the setpoint as expected. In contrast, the cyan solid line labeled as *Nominal* shows the nominal parameter baseline, which does not reach the setpoint over the 10 s simulation horizon. This suggests that, under parameter uncertainty, the fixed-gain approach is sensitive to parameter mismatch and can suffer performance degradation.

Fig. 3 shows the outputs of the local reference dynamics $C\hat{x}_{r,i}(t)$ and the adaptive gains during payload transportation. The top panel shows the components of $C\hat{x}_{r,i}(t)$ for $i = 1, \ldots, 4$. The bottom-left panel shows all four entries of each $K_i(t) \in \mathbb{R}^{1\times 4}$, and the bottom-right panel shows the scalar gain $L_i(t)$ for each robot. The local reference outputs rapidly synchronize and approach the setpoint by 2 s, consistent with the predicted synchronization and reference convergence in Lemma 1. In addition, the adaptive gains for all robots remain bounded throughout the simulation, avoiding excessively large control inputs and confirming the stability guarantees of Theorem 1.

VI. CONCLUSION

This paper proposed a distributed model reference adaptive control framework for uncertain multi-channel linear plants. Using only local channel information, networked agents collectively generate an emergent reference model and make the plant track the resulting reference trajectory via distributed MRAC, without centralized coordination or exact global parameter knowledge. We proved asymptotic convergence to the setpoint with bounded adaptive gains, and simulations confirmed adaptive tracking performance under parametric uncertainty. Future work includes extending the current framework to time-varying reference tracking and output-feedback settings, as it is currently limited to constant setpoint regulation under full-state measurement.

APPENDIX

We provide the proof of Lemma 1. Let $\mathsf{L} = [\mathsf{l}_{ij}] \in \mathbb{R}^{N\times N}$ denote the Laplacian matrix of an undirected, connected, unit-weight graph, where $\mathsf{l}_{ij} := -1$ if $(i,j) \in \mathscr{E}$ and $\mathsf{l}_{ij} := 0$ otherwise for $i \neq j$, and $\mathsf{l}_{ii} := -\sum_{j\neq i} \mathsf{l}_{ij}$. It satisfies $\mathsf{L} = \mathsf{L}^\top \succeq 0$, $\mathsf{L}1_N = 0$, and $\lambda_2(\mathsf{L}) > 0$. Moreover, there exists $R \in \mathbb{R}^{N\times(N-1)}$ such that $1_N^\top R = 0$, $R^\top R = I_{N-1}$, and $R^\top \mathsf{L} R = \Lambda$, where $\Lambda := \mathrm{diag}(\lambda_2(\mathsf{L}),\ldots,\lambda_N(\mathsf{L})) \succ 0$.

With these definitions, write the local reference dynamics (8) in closed-loop form as

$$\begin{aligned}\dot{\hat{x}}_{g,i}(t) &= H_i(t)\hat{x}_{g,i}(t) + Fr \\ &\quad + k_g \sum_{j\in\mathscr{N}_i} (\hat{x}_{g,j}(t) - \hat{x}_{g,i}(t)) + k_g \sum_{j\in\mathscr{N}_i} (\hat{z}_{g,j}(t) - \hat{z}_{g,i}(t)), \\ \dot{\hat{z}}_{g,i}(t) &= -k_g \sum_{j\in\mathscr{N}_i} (\hat{x}_{g,j}(t) - \hat{x}_{g,i}(t)),\end{aligned}$$

where $H_i(t) := \bar{A}_M - N\bar{b}_{M,i}G_i(t) \in \mathbb{R}^{\bar{n}\times\bar{n}}$. Define the column stack of $\hat{x}_{g,i}$ and $\hat{z}_{g,i}$ as $\rho := [\hat{x}_{g,1}^\top \cdots \hat{x}_{g,N}^\top]^\top$ and $\zeta := [\hat{z}_{g,1}^\top \cdots \hat{z}_{g,N}^\top]^\top$, respectively. Then, using $\sum_{j\in\mathscr{N}_i}(x_j - x_i) = -(\mathsf{L}x)_i$, the dynamics can be written in compact form as

$$\begin{aligned}\dot{\rho}(t) &= \mathrm{diag}_N(H_i(t))\rho(t) + (1_N \otimes I_{\bar{n}})Fr \\ &\quad - k_g(\mathsf{L}\otimes I_{\bar{n}})\rho(t) - k_g(\mathsf{L}\otimes I_{\bar{n}})\zeta(t) \\ \dot{\zeta}(t) &= k_g(\mathsf{L}\otimes I_{\bar{n}})\rho(t).\end{aligned}$$

Now, apply the coordinate change

$$\begin{bmatrix}\rho_{\mathrm{avg}} \\ \rho_{\mathrm{dev}}\end{bmatrix} := \begin{bmatrix}\frac{1}{N}1_N^\top \otimes I_{\bar{n}} \\ R^\top \otimes I_{\bar{n}}\end{bmatrix}\rho, \quad \begin{bmatrix}\zeta_{\mathrm{avg}} \\ \zeta_{\mathrm{dev}}\end{bmatrix} := \begin{bmatrix}\frac{1}{N}1_N^\top \otimes I_{\bar{n}} \\ R^\top \otimes I_{\bar{n}}\end{bmatrix}\zeta. \quad (22)$$

Moreover, define

$$A_{\mathrm{cl}} := \bar{A}_M - \bar{B}_M G^*, \quad \widetilde{G}_i(t) := G_i(t) - G_i^*.$$

Note that $\rho = (1_N \otimes I_{\bar{n}})\rho_{\mathrm{avg}} + (R\otimes I_{\bar{n}})\rho_{\mathrm{dev}}$, $\zeta = (1_N \otimes I_{\bar{n}})\zeta_{\mathrm{avg}} + (R\otimes I_{\bar{n}})\zeta_{\mathrm{dev}}$, and A_{cl} is Hurwitz. Define the nominal and perturbation parts of the local closed-loop matrix as

$$\begin{aligned}H_i^* &:= \bar{A}_M - N\bar{b}_{M,i}G_i^*, \\ \widetilde{H}_i(t) &:= H_i(t) - H_i^* = -N\bar{b}_{M,i}\widetilde{G}_i(t).\end{aligned}$$

Then, the dynamics of the stacked variables become

$$\begin{aligned}\dot{\rho}_{\mathrm{avg}}(t) &= A_{\mathrm{cl}}\rho_{\mathrm{avg}}(t) + Fr - \left(\sum_{i=1}^N \bar{b}_{M,i}\widetilde{G}_i(t)\right)\rho_{\mathrm{avg}}(t) \\ &\quad + (\frac{1}{N}1_N^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i(t))(R\otimes I_{\bar{n}})\rho_{\mathrm{dev}}(t) \\ \dot{\rho}_{\mathrm{dev}}(t) &= (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i(t))(1_N\otimes I_{\bar{n}})\rho_{\mathrm{avg}}(t) \\ &\quad + (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i(t))(R\otimes I_{\bar{n}})\rho_{\mathrm{dev}}(t) \\ &\quad - k_g(\Lambda\otimes I_{\bar{n}})\rho_{\mathrm{dev}}(t) - k_g(\Lambda\otimes I_{\bar{n}})\zeta_{\mathrm{dev}}(t),\end{aligned}$$

and $\dot{\zeta}_{\mathrm{avg}}(t) = 0$, $\dot{\zeta}_{\mathrm{dev}}(t) = k_g(\Lambda\otimes I_{\bar{n}})\rho_{\mathrm{dev}}(t)$.

Since A_{cl} is Hurwitz, there exists a unique ρ_{avg}^* such that $A_{\mathrm{cl}}\rho_{\mathrm{avg}}^* + Fr = 0$, where $\rho_{\mathrm{avg}}^* = x_g^*$ by the uniqueness of the steady state. Now, we want to show that ρ_{avg} converges to ρ_{avg}^*. For this, set $\eta := \rho_{\mathrm{avg}} - \rho_{\mathrm{avg}}^*$ and $\xi := \zeta_{\mathrm{dev}} - \zeta_{\mathrm{dev}}^*$ where

$$\zeta_{\mathrm{dev}}^* = \frac{1}{k_g}(\Lambda^{-1}\otimes I_{\bar{n}})(R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i^*)(1_N\otimes I_{\bar{n}})\rho_{\mathrm{avg}}^*.$$

In addition, define

$$\begin{aligned}M_1 &:= (\frac{1}{N}1_N^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i^*)(R\otimes I_{\bar{n}}), \\ M_2 &:= (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i^*)(1_N\otimes I_{\bar{n}}), \\ M_3 &:= (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(H_i^*)(R\otimes I_{\bar{n}}), \\ \widetilde{M}_1(t) &:= \left(\frac{1}{N}1_N^\top \otimes I_{\bar{n}}\right)\mathrm{diag}_N(\widetilde{H}_i(t))(R\otimes I_{\bar{n}}), \\ \widetilde{M}_2(t) &:= (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(\widetilde{H}_i(t))(1_N\otimes I_{\bar{n}}), \\ \widetilde{M}_3(t) &:= (R^\top \otimes I_{\bar{n}})\,\mathrm{diag}_N(\widetilde{H}_i(t))(R\otimes I_{\bar{n}}), \\ \widetilde{\Delta}_a(t) &:= -\left(\sum_{i=1}^N \bar{b}_{M,i}\widetilde{G}_i(t)\right)(\eta(t) + \rho_{\mathrm{avg}}^*) + \widetilde{M}_1(t)\rho_{\mathrm{dev}}(t), \\ \widetilde{\Delta}_b(t) &:= \widetilde{M}_2(t)(\eta(t) + \rho_{\mathrm{avg}}^*) + \widetilde{M}_3(t)\rho_{\mathrm{dev}}(t).\end{aligned}$$

Then, the error dynamics become

$$\dot{\psi}(t) = A_{k_g}\psi(t) + \Delta_\psi(t), \quad (23)$$

where $\psi := [\eta^\top\ \rho_{\mathrm{dev}}^\top\ \xi^\top]^\top$, $\Delta_\psi := [\widetilde{\Delta}_a^\top\ \widetilde{\Delta}_b^\top\ 0_{(N-1)\bar{n}}^\top]^\top$, and

$$A_{k_g} := \begin{bmatrix} A_{\mathrm{cl}} & M_1 & 0_{\bar{n}\times(N-1)\bar{n}} \\ M_2 & M_3 - k_g(\Lambda\otimes I_{\bar{n}}) & -k_g(\Lambda\otimes I_{\bar{n}}) \\ 0_{(N-1)\bar{n}\times\bar{n}} & k_g(\Lambda\otimes I_{\bar{n}}) & 0_{(N-1)\bar{n}\times(N-1)\bar{n}}\end{bmatrix}.$$

To bound $\Delta_\psi(t)$, define $\delta_\Delta(t) := \max_{i\in\mathscr{N}} ||\widetilde{G}_i(t)||$. From the definitions of $\widetilde{H}_i(t)$, $\widetilde{M}_k(t)$ for $k = 1,2,3$, $\widetilde{\Delta}_a(t)$, and $\widetilde{\Delta}_b(t)$, there exists $c_\Delta > 0$ such that

$$|\widetilde{\Delta}_a(t)| + |\widetilde{\Delta}_b(t)| \le c_\Delta \delta_\Delta(t)\big(1 + |\psi(t)|\big). \quad (24)$$

Moreover, $\delta_\Delta(t)$ converges to 0 exponentially because $G_i(t)$ converges to G_i^* exponentially by assumption.

Since A_{cl} is Hurwitz, choose $Q \succ 0$ and let $P_V \succ 0$ be the unique solution of $A_{\mathrm{cl}}^\top P_V + P_V A_{\mathrm{cl}} = -Q$. Consider $V(\psi) := V_1(\eta) + V_2(\rho_{\mathrm{dev}},\xi)$ where $V_1(\eta) := (\eta^\top P_V \eta)/2$ and

$$V_2(\rho_{\mathrm{dev}},\xi) := \frac{1}{2}\rho_{\mathrm{dev}}^\top(\Lambda\otimes I_{\bar{n}})\rho_{\mathrm{dev}} + \frac{1}{2}\xi^\top(\Lambda\otimes I_{\bar{n}})\xi. \quad (25)$$

Along (23),

$$\begin{aligned}\dot{V}_1 &= -\frac{1}{2}\eta^\top Q\eta + \eta^\top P_V M_1\rho_{\text{dev}} + \eta^\top P_V\widetilde{\Delta}_a \\ &\le -\lambda_q|\eta|^2 + c_{M_1}|\eta||\rho_{\text{dev}}| + ||P_V|||\eta||\widetilde{\Delta}_a|, \end{aligned} \tag{26}$$

where $\lambda_q := \lambda_{\min}(Q)/2 > 0$ and $c_{M_1} := ||P_V M_1||$.

Next,

$$\begin{aligned}\dot{V}_2 &= \rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})M_2\eta + \rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})M_3\rho_{\text{dev}} \\ &\quad - k_g\rho_{\text{dev}}^\top(\Lambda^2\otimes I_{\bar{n}})\rho_{\text{dev}} + \rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})\widetilde{\Delta}_b.\end{aligned}$$

The first term is bounded as $\rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})M_2\eta \le c_{M_2}|\rho_{\text{dev}}||\eta|$, with $c_{M_2} := ||(\Lambda\otimes I_{\bar{n}})M_2||$. By the generalized Rayleigh quotient, we obtain $\rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})M_3\rho_{\text{dev}} \le \mu\rho_{\text{dev}}^\top(\Lambda^2\otimes I_{\bar{n}})\rho_{\text{dev}}$, where

$$\mu := \frac{1}{2}\lambda_{\max}\big(M_3(\Lambda^{-1}\otimes I_{\bar{n}}) + (\Lambda^{-1}\otimes I_{\bar{n}})M_3^\top\big). \tag{27}$$

The third term is $\rho_{\text{dev}}^\top(\Lambda^2\otimes I_{\bar{n}})\rho_{\text{dev}} \ge (\lambda_2(\mathsf{L}))^2|\rho_{\text{dev}}|^2$. Finally, $\rho_{\text{dev}}^\top(\Lambda\otimes I_{\bar{n}})\widetilde{\Delta}_b \le ||\Lambda|||\rho_{\text{dev}}||\widetilde{\Delta}_b|$. Combining (26) and the above bounds yields

$$\begin{aligned}\dot{V} &\le -\lambda_q|\eta|^2 - (k_g-\mu)(\lambda_2(\mathsf{L}))^2|\rho_{\text{dev}}|^2 + c_M|\eta||\rho_{\text{dev}}| \\ &\quad + c_P|\psi|\big(|\widetilde{\Delta}_a| + |\widetilde{\Delta}_b|\big),\end{aligned}$$

for some $c_M, c_P > 0$. Using Young's inequality, $c_M|\eta||\rho_{\text{dev}}| \le (\lambda_q|\eta|^2)/2 + (c_M|\rho_{\text{dev}}|)^2/(2\lambda_q)$.

Hence,

$$\begin{aligned}\dot{V} &\le -\frac{\lambda_q}{2}|\eta|^2 - \Big((k_g-\mu)(\lambda_2(\mathsf{L}))^2 - \frac{c_M^2}{2\lambda_q}\Big)|\rho_{\text{dev}}|^2 \\ &\quad + c_P|\psi|\big(|\widetilde{\Delta}_a| + |\widetilde{\Delta}_b|\big).\end{aligned}$$

Therefore, for any

$$k_g > k_g^* := \mu + \frac{c_M^2}{2\lambda_q(\lambda_2(\mathsf{L}))^2},$$

there exists $c_\mu > 0$ such that

$$\dot{V} \le -\frac{\lambda_q}{2}|\eta|^2 - c_\mu|\rho_{\text{dev}}|^2 + c_P|\psi|\big(|\widetilde{\Delta}_a| + |\widetilde{\Delta}_b|\big). \tag{28}$$

Next, using $|\psi| \le (1+|\psi|^2)/2$ and (24), the last term in (28) is bounded by $c_1\delta_\Delta(t)(1+|\psi|^2)$ for some constant $c_1 > 0$. Since $V(\psi)$ is a positive definite quadratic form, there exists a constant $c_2 > 0$ such that $1+|\psi|^2 \le c_2(1+V)$. Combining these facts yields

$$\dot{V} \le c_3\delta_\Delta(t)(1+V)$$

for some $c_3 > 0$. Applying Gronwall's inequality to $1+V(t)$ gives

$$1+V(t) \le (1+V(0))\exp\left(c_3\int_0^t \delta_\Delta(s)\,ds\right).$$

The integral is finite because $\delta_\Delta(t)$ decays exponentially. Consequently, $V(t)$ and $\psi(t)$ are bounded for all $t \ge 0$. Since $\psi(t)$ is bounded, (24) implies $|\Delta_\psi(t)| \le c_4\delta_\Delta(t)$ for some $c_4 > 0$, and hence $|\Delta_\psi(t)|$ decays exponentially.

Consider now the nominal system

$$\dot{\psi}(t) = A_{k_g}\psi(t),$$

which is obtained from (23) by setting $\Delta_\psi(t) \equiv 0$. Along the nominal system, (28) yields

$$\dot{V} \le -(\lambda_q/2)|\eta|^2 - c_\mu|\rho_{\text{dev}}|^2.$$

Hence, $\dot{V} = 0$ implies $\eta = 0$ and $\rho_{\text{dev}} = 0$. On this set, $\dot{\xi} = k_g(\Lambda\otimes I_{\bar{n}})\rho_{\text{dev}} = 0$. Moreover, invariance of $\rho_{\text{dev}} = 0$ requires $k_g(\Lambda\otimes I_{\bar{n}})\xi = 0$. Since $\Lambda \succ 0$, it follows that $\xi = 0$. Therefore, the largest invariant set contained in $\{\dot{V} = 0\}$ is the origin. Since V is positive definite and radially unbounded, LaSalle's invariance principle implies that the origin of the nominal system is globally asymptotically stable. Since the nominal system is LTI, A_{k_g} is Hurwitz for all $k_g > k_g^*$.

Therefore, the actual error system (23) is a Hurwitz linear system driven by an exponentially decaying perturbation $\Delta_\psi(t)$, so that $\psi(t)$ converges to the origin exponentially for all $k_g > k_g^*$. Consequently, $\eta(t)$, $\rho_{\text{dev}}(t)$, and $\xi(t)$ all converge to the origin exponentially. Recalling that $\rho = (1_N\otimes I_{\bar{n}})\rho_{\text{avg}} + (R\otimes I_{\bar{n}})\rho_{\text{dev}}$ and $\rho_{\text{avg}} = \eta + \rho_{\text{avg}}^*$, we conclude that $\hat{x}_{g,i}(t)$ converges to $\rho_{\text{avg}}^* = x_g^*$ exponentially for all i. Similarly, from $\zeta = (1_N\otimes I_{\bar{n}})\zeta_{\text{avg}} + (R\otimes I_{\bar{n}})\zeta_{\text{dev}}$, $\zeta_{\text{avg}} = 0$, and $\zeta_{\text{dev}} = \xi + \zeta_{\text{dev}}^*$, it follows that $\hat{z}_{g,i}(t)$ remains bounded for all i.

Since $\eta(t) = \rho_{\text{avg}}(t) - \rho_{\text{avg}}^*$ converges to zero exponentially, $\rho_{\text{avg}}(t)$ converges exponentially to $\rho_{\text{avg}}^* = x_g^*$. By (11), $x_r(t)$ converges exponentially to x_r^*. □

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